

# String rearrangement of gauge theories

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## Abstract

Feynman diagram expressions in ordinary field theories can be written in a string-like manner. The methods and the advantages for doing so are briefly discussed.

## 1. Introduction

There are three reasons why one wants to arrange ordinary Feynman diagrams in a string-like manner: it simplifies calculations and it gives new insights into gauge and gravitational theories. Moreover, this is done (graphically) all the times so we might as well find out exactly what that means. For example, The tree diagrams for the process  $\pi^+ K^0 \rightarrow \pi^0 K^+$  are given by Figs. 1(a) and (b), but one often shows only the quark diagram Fig. 1(c), which can be considered also as a string diagram with hadronic strings strung between  $q\bar{q}$  pairs. String diagrams are also used for pure QCD in the large- $N_c$  limit as shown in Fig. 2.

Is it possible then to make the string-like diagrams *quantitative* by writing down for them a set of ‘Feynman rules’? When one tries to do that a number of problems is encountered. For ordinary Feynman diagrams, (i) loop momenta  $k_a$  have to be introduced and integrated over; (ii) interactions occur at the vertices though particles propagate freely between them. In particular, loop momenta  $k_a$  injected at the vertices can change the direction of the combined momentum flow. Similarly, flavour, colour, and spin are altered at the vertices; (iii) gauge invariance determines the vertex factor for gauge interactions, for example to be  $\epsilon(p) \cdot (q' + q'')$  in scalar QED, where  $p$  is the momentum of the external photon and  $q', q''$  are the charged particle momenta; (iv) a sum of many Feynman diagrams is needed to describe a process in a given order. In contrast, for

**Figure 1.** Lowest-order diagrams for  $\pi^+ K^0 \rightarrow \pi^0 K^+$ .

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## 2. Momentum, flavour, colour, and spin flows

To convert (i) to (i'), it is sufficient to introduce a Schwinger parameter  $\alpha_r$  for the denominator of every propagator:

$$(-q_r^2 + m_r^2 - i\epsilon)^{-1} = i \int_0^\infty d\alpha_r \exp[-i\alpha_r(m_r^2 - q_r^2)] . \quad (1)$$

Substituting this into the general expression for a T-matrix amplitude

$$T(p) = \int \left( \prod_a d^4 k_a \right) S_0(q, p) \prod_r (-q_r^2 + m_r^2 - i\epsilon)^{-1} , \quad (2)$$

the loop momentum integrations can be explicitly carried out to yield [2]

$$T(p) \sim \int_0^\infty \left( \prod_r d\alpha_r \right) \Delta^{-2}(\alpha) S(q, p) \prod_r \exp[-i\alpha_r(m_r^2 - q_r^2)] . \quad (3)$$

**Figure 2.** A pure-gluon QCD process.

string diagrams, (i') if we consider only hadronic *ground* states so as to freeze string excitations, the propagation of a string is described only by the 'proper time' parameter  $\tau$ , so that scattering amplitudes are expressed as integrations over the proper-time parameters  $\tau_m$ ; (ii') the geometrical shape of a string diagram is arbitrary (reparametrization and conformal invariance), so vertices cannot be present and strings propagate freely throughout the diagram. In particular, external momenta can merge and divide inside a diagram but there are no additional loop momenta present to alter their directions. Flavour and colour flow smoothly along the quark lines as given respectively by Fig. 1(c) and Fig. 2(b), and similarly for spin as we shall see later; (iii') conformal invariance determines external gauge-interaction vertices, for example to be  $\epsilon(p) \cdot [\partial_\tau x(\tau)] \exp[ip \cdot x(\tau)]$  for scalar QED where  $x^\mu(\tau)$  is the spacetime (operator) coordinates of the string at the proper time  $\tau$ . The vertex factor for this operator is then  $\epsilon(p) \cdot [\partial_\tau(-i\partial/\partial p)]$ , quite different from the corresponding field-theoretic vertex in (iii); (iv') (Veneziano) duality [1] is valid. The sum of many Feynman diagrams is replaced by one or few string diagrams, as in Fig. 1.

In the rest of this note, we will show how to rearrange (i)–(iv) to resemble (i')–(iv'). The calculational and conceptual advantages for doing so will also be briefly discussed.

This is the string-like form where momentum flow is described by  $q_r$ , a quantity best thought of as the current flowing through the  $r$ th line of an electric circuit given by the Feynman diagram, in which the branch resistances are  $\alpha_s$  and the external currents are  $p_i$ . Explicit rules are available to calculate these currents and other quantities in  $T(p)$  directly from the Feynman diagram. If a proper time  $\tau$  is assigned to each vertex, and if line  $r = (ij)$  connects vertices  $j$  to  $i$ , then  $\alpha_r$  can also be interpreted as the proper-time difference  $|\tau_i - \tau_j|$ .

The factor  $S_0(q, p)$  in (2) contains all the vertex factors and numerators of propagators, so it encodes the flows of flavour, colour, and spin and is given by a sum of products of these factors. The quantity  $S(q, p)$  in (3) is equal to  $S_0 + S_1 + S_2 + \dots$ , and can be obtained from  $S_0$  through momentum contraction [2]. The factors for flavour, colour, and spin flows can all be read off directly from the appropriate quark, or string-like, diagrams, and it is important to note that the quark diagrams are *different for different flows*. The general rules and their derivations are given elsewhere [3], but specific examples can be seen from Figs. 3 and 4 respectively for colour and spin flows. In each case diagram (a) is the Feynman diagram and diagram (b) is the equivalent string-like diagram from which the factors for  $S_0$  can be read off to be

$$S_0^{(m)colour} = T^a T^b T^c T^d \text{tr}(T^e T^f T^g T^h) , \quad (4)$$

$$S_0^{spin} = [p_2 q_1] \langle q_1 p_3 \rangle \cdot [p_4 p_5] \cdot \langle k_5 q_9 \rangle [q_9 q_3] \langle q_3 p_1 \rangle \cdot \langle p_6 q_6 \rangle [q_6 q_5] \langle q_5 q_8 \rangle [q_8 k_6] . \quad (5)$$

can be obtained from other string-like diagrams, which in the case of  $U(N)$  can be obtained from 3(b) by crossing the quark lines at the vertices. For  $SU(N)$  there are other quark diagrams in which some of the internal gluon lines are omitted. The spin-flow (or more correctly helicity-flow) factor  $S_0^{spin}$  in (5) is obtained by assuming the fermion masses to be zero. In that case the string-like diagram 4(b) is unique once 4(a) is given, and the direction the fermion lines turn depends on the helicities of the external particles, indicated in the diagram by a  $+$  or a  $-$  sign. More diagrams will be necessary if the internal particles are massive. The square and angular brackets are the overlap of the massless Dirac wave functions with definite helicities:

$$[p_i p_j] = \bar{u}_+(p_i) u_-(p_j), \quad \langle p_i p_j \rangle = \bar{u}_-(p_i) u_+(p_j), \quad (6)$$

When internal momenta  $q_r$  appear in these brackets, it is understood that they should first be expanded in terms of the external massless momenta  $p_i$  with only diagonal terms kept. Note that Dirac matrices are completely absent so the usual four-channel problem is reduced to a one-channel expression. This is possible because of helicity and chirality conservations for massless fermions. Note that it would have been impossible to write (5) before the loop momenta were eliminated in going from (2) to (3).

The final factor  $S_0$  is obtained from  $S_0^{(m)colour} S_0^{spin} \dots$  by summing over all  $m$ . The main advantage in this string-like arrangement is to be able to use the spinor helicity technique [4], developed originally for *tree* diagrams, now for arbitrary processes with any number of loops. It also allows the gauge-invariant colour subamplitudes to be easily separated.

### 3. External gauge vertices

The scalar QED vertex (*iii*) can be replaced by the string-like vertex (*iii'*) by using differential circuit identities [3]. The presence of  $\partial/\partial\tau$  in the latter expression allows integration-by-parts to be used, to redistribute the gauge-dependent terms among different diagrams in order to minimize their appearance and thereby increase computability. It also makes the Ward-Takahashi identity realized in a different way, and allows the possibility of formulating gauge invariance in another way.

### 4. Duality

Veneziano duality (*iv'*) can be simulated by formally summing up a number of Feynman diagrams into a single integral expression, at least for QED-like theories [5]. Take for example the sum of the  $6!4!3!$  QED diagrams, obtained from Fig. 5 by permuting the photon vertices along each of the three charged lines.

**Figure 3.** An example to illustrate colour flow. See eq. (4).

**Figure 4.** An example to illustrate spin flow. See eq. (5).

$T^a$  in (5) are the  $(S)U(N)$  generators in the fundamental representation. The superscript  $(m)$  indicates that there are many colour factors for Fig. 3(a), and that in (4) is the one appropriate to Fig. 3(b). Other colour factors

**Figure 5.** A QED diagram.

In the proper-time representation (3), a proper time  $[\tau'_i(1 \leq i \leq 6), \tau''_j(1 \leq j \leq 4), \tau'''_k(1 \leq k \leq 3)]$  is assigned to each vertex, and each Feynman diagram is given by a fixed ordering of the  $\tau'$ 's,  $\tau''$ 's, and  $\tau'''$ 's, so for each diagram the proper-time integration region in (3) is the product of three *hyper-triangular regions*. The ‘dual sum’ of these  $6!4!3!$  diagrams is obtained by summing all these permutations, and is thus given by a *single* integral over the product of three *hypercube* regions, one each for  $\tau', \tau''$ , and  $\tau'''$ . Moreover, it is gauge invariant. Unfortunately, it is often impossible to carry out this single integral analytically, but it can be used as a starting point for gauge-invariant approximations. For example, the result for the soft-photon eikonal approximation can be obtained very easily in this way.

It is also worthwhile pointing out an interesting parallel. Each Feynman diagram with  $n$  vertices is a sum of  $n!$  time-ordered (old-fashioned) diagrams. An individual time-ordered diagram is not Lorentz invariant, but the Feynman diagram is. In our case, the dual sum puts together Feynman diagrams that have different *proper-time* orderings, each of which is not gauge invariant, but the dual sum is.

## 5. References

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